

ANALYTICITY OF SEMIGROUPS ON THE RIGHT HALF-PLANE

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ABSTRACT. This paper is devoted to the study of semigroups of composition operators and semigroups of holomorphic mappings. We establish conditions under which these semigroups can be extended in their parameter to sector given a priori. We show that the size of this sector can be controlled by the image properties of the infinitesimal generator, or, equivalently, by the geometry of the so-called associated planar domain. We also give a complete characterization of all composition operators acting on the Hardy space H^p on the right half-plane.

Key words and phrases: holomorphic mapping, semigroup, composition operator, Hardy space, function convex in one direction.

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1. INTRODUCTION

In this paper, we study semigroups of holomorphic self-mappings of the right half-plane $\Pi = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ and semigroups of bounded linear operators acting on the Hardy spaces $H^p(\Pi)$, $p > 1$, and their interaction.

Recall that a one-parameter family $\mathbf{T} := \{T(t), t \geq 0\}$ of bounded linear operators on a Banach space X is said to be a C_0 -semigroup if it satisfies

- (a) $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$,
- (b) $\lim_{t \rightarrow 0^+} T(t) = T(0) = I$, where I is the identity operator on X .

For each C_0 -semigroup \mathbf{T} on X , there exist scalars $a \geq 0$ and $M \geq 1$ such that $\|T(t)\| \leq Me^{at}$ for all $t \geq 0$. If $\|T(t)\| \leq 1$ for all $t \geq 0$, then the \mathbf{T} is said to be a contractive semigroup.

Denote $\mathcal{D} := \left\{ x \in X : \lim_{t \rightarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\}$. The linear operator Γ defined on \mathcal{D} by

$$\Gamma x := \lim_{t \rightarrow 0} \frac{T(t)x - x}{t} = \frac{dT(t)x}{dt}$$

is called the *infinitesimal generator* of the semigroup \mathbf{T} . The domain \mathcal{D} is a dense linear subspace of X (see, for example [23]).

Let X^* be the dual space to X . Recall that a linear operator Γ (in general, unbounded) defined on a dense subspace $\mathcal{D} \subset X$ is called *dissipative* if for all $x \in \mathcal{D}$, $\|x\| = 1$, there exists $x^* \in X^*$ with $\langle x, x^* \rangle = 1$ which satisfies

$$\operatorname{Re}\langle \Gamma x, x^* \rangle \leq 0.$$

The functional x^* , is not unique in general (see, for example, [23]); it is often called a support functional at the point x .

The classical Lummer–Phillips theorem (see [23], [24]) asserts that *an operator Γ generates a contractive semigroup if and only if Γ is dissipative and $\lambda I - \Gamma$ is surjective for some (hence for all) $\lambda > 0$* . Moreover, if $\lambda I - \Gamma$ is surjective for every $\lambda > 0$, then $\mu I - \Gamma$ is surjective for every $\mu \in \mathbb{C}$ such that $\operatorname{Re} \mu > 0$ (see, for example, [23, Remark 5.4]).

An important class of bounded linear operators which links to Complex Analysis and attracts special attention consists of composition operators. Each holomorphic self-mapping $F \in \operatorname{Hol}(D)$ of a domain $D \subset \mathbb{C}$ induces a composition operator C_F defined by

$$C_F : \varphi \mapsto \varphi \circ F$$

on the Frechét space $\operatorname{Hol}(D, \mathbb{C})$. Composition operators make up a wide class of explicit examples in Operator Theory. Quoting [12], “...they are surprisingly general and occur in settings other than the obvious ones.” In order to consider composition operators as a part of Operator Theory, it is necessary to study properties of their restrictions to different Banach spaces $X \subset \operatorname{Hol}(D, \mathbb{C})$.

We proceed by reviewing several concepts and facts related to semigroups of holomorphic self-mappings of a domain $D \subset \mathbb{C}$. Throughout this paper, we denote the set of holomorphic functions on D taking values in a set U by $\operatorname{Hol}(D, U)$ and let $\operatorname{Hol}(D) := \operatorname{Hol}(D, D)$.

A *one-parameter continuous semigroup* (semigroup, for short) on D is a family $\mathbf{F} = \{F_t\}_{t \geq 0} \subset \operatorname{Hol}(D)$ such that

- (i) $F_t(F_s(z)) = F_{t+s}(z)$ for all $t, s \geq 0$ and $z \in D$,
- (ii) $\lim_{t \rightarrow 0^+} F_t(z) = z$ for all $z \in D$.

It follows by [5] that each semigroup on a simply connected domain D is differentiable with respect to $t \in \mathbb{R}^+ = [0, \infty)$. Thus, for each one-parameter continuous semigroup, the limit

$$\lim_{t \rightarrow 0^+} \frac{F_t(z) - z}{t} := f(z), \quad z \in D,$$

exists and defines a holomorphic function $f \in \operatorname{Hol}(D, \mathbb{C})$ called the (*infinitesimal*) *generator of \mathbf{F}* . Moreover, the function $u : \mathbb{R}^+ \times D \rightarrow \mathbb{C}$ defined by

$u(t, z) := F_t(z)$, is the unique solution of the Cauchy problem

$$\begin{cases} \frac{\partial u(t, z)}{\partial t} = f(u(t, z)), \\ u(0, z) = z, \quad z \in D. \end{cases} \quad (1.1)$$

Since all simply connected domains are holomorphically equivalent, one usually considers semigroups on either the right half-plane Π , or the open unit disk \mathbb{D} (see [26]). It is well known that for a semigroup \mathbf{F} on Π that contains neither the identity nor an elliptic automorphism, there exists a unique $\tau \in \overline{\Pi} \cup \{\infty\}$, called the Denjoy–Wolff point of \mathbf{F} , to which $F_t(z)$ converges as $t \rightarrow \infty$ for all $z \in \Pi$ (see [26] for details).

In this paper, we study semigroups on the right half-plane Π with the Denjoy–Wolff point at ∞ . The generators of such semigroups satisfy $\operatorname{Re} f(z) \geq 0$; and by Julia’s Lemma, $\operatorname{Re} F_t(z)$ is a non-decreasing function in t for $t \geq 0$ (see, for example, [26]).

Clearly, each semigroup $\mathbf{F} = \{F_t\}_{t \geq 0} \subset \operatorname{Hol}(D)$ induces the semigroup $\mathbf{C} = \{C_t, t \geq 0\}$ of composition operators defined by

$$C_F : \varphi \mapsto \varphi \circ F_t \quad (1.2)$$

on the Frechét space $\operatorname{Hol}(D, \mathbb{C})$. Although composition operators have a long history and diverse applications (see, for example, [12] and [25] and references therein), interest in semigroups of composition operators essentially began with the paper [5] by Berkson and Porta and the review [27] by Siskakis. During the last two decades, various properties of semigroups of composition operators have been studied by many mathematicians for different Banach spaces $X \subset \operatorname{Hol}(D, \mathbb{C})$ (such as the Dirichlet space, the Hardy spaces, VMOA etc.) in the case $D = \mathbb{D}$, the open unit disk; see, for example, [25, 27, 3, 7]. In particular, the problem of finding conditions for analytic extension of semigroups of composition operators mentioned above was considered in [4, 16].

However, relatively little is known about semigroups of composition operators on Banach spaces of holomorphic functions in the right half-plane Π . It turns out that properties of even single composition operators on Banach spaces of holomorphic functions on Π are different from analogous properties of composition operators on \mathbb{D} . In particular, each composition operator on the Hardy space $H^p(\mathbb{D})$, is bounded, but the composition operator C_F on $H^p(\Pi)$ is bounded if and only if the inducing function F has a finite angular derivative at infinity; see [19]. Also, there is no compact composition operator on $H^p(\Pi)$; see [22] and [11].

The paper is organized as follows. Section 2 contains auxiliary results that are partially equivalent to some known statements obtained in differing settings. We prove them for the sake of completeness.

Sections 3 and 4 contain the main results. Namely, in Section 3, we study semigroups of holomorphic self-mappings of Π whose generators have argument controlled by functions given a priori. We find sectors to which such a semigroup $\mathbf{F} = \{F_t\}_{t \geq 0}$ as well as its restrictions to invariant subsets can be analytically extended in the semigroup parameter. We also describe domains (depending on $z \in \Pi$) which contain the semigroup trajectory $\{F_t(z), t \geq 0\}$.

In Section 4, we deal with semigroups of composition operators on the Hardy space $H^p(\Pi)$, $p \in (1, \infty)$. Recall that

$$H^p(\Pi) = \left\{ \varphi \in \text{Hol}(\Pi, \mathbb{C}) : \|\varphi\| = \sup_{x > 0} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} |\varphi(x + iy)|^p dy \right)^{1/p} < \infty \right\}.$$

Thus, we amplify results obtained earlier in [2] and [4]. For a given semigroup of composition operators on $H^p(\Pi)$, the main result describes its analytic extension to a sector. Roughly speaking, principal parts of Theorems 3.1 and 4.3 can be combined as follows. For an open sector $\Omega \subset \mathbb{C}$ with vertex at zero, let $\tilde{\Omega} = \{w \in \mathbb{C} : \text{Re}(zw) > 0 \text{ for all } z \in \Omega\}$. We prove that under some mild conditions on an infinitesimal generator $f \in \text{Hol}(\Pi)$ and a sector Ω , the following assertions are equivalent:

- the image $f(\Pi)$ is contained in $\tilde{\Omega}$;
- the semigroup \mathbf{F} generated by f can be analytically extended in its parameter to the sector Ω ;
- the conformal mapping onto the planar domain (associated with \mathbf{F}) is convex in every direction in Ω ;
- the semigroup \mathbf{T} of composition operators induced by \mathbf{F} can be analytically extended in its parameter to the sector Ω ;
- for each $\zeta \in \Omega$, the linear operator $\zeta f \frac{\partial}{\partial z}$ generates a contractive semigroup.

Additionally, Theorem 4.2, gives a criterion for a bounded linear operator acting on $H^p(\Pi)$ to be a composition operator.

We mention again an essential difference between semigroups of composition operators on Hardy spaces on \mathbb{D} and those on Hardy spaces on Π . Semigroups preserving the origin in \mathbb{D} as well as induced composition semigroups on Hardy spaces were considered in [16]. One of the results of that work implies that any semigroup of holomorphic self-mappings restricted to invariant subsets (which are disks of radii $r < 1$) can be analytically extended to a sector, and the sector is wider the smaller r is. Consequently, the induced composition semigroup can be extended to a wider sector too. However, each

analytic semigroup of bounded composition operators on $H^p(\Pi)$ is induced by a semigroup of parabolic type; see Corollary 4.3. In addition, as we show, the restriction of a semigroup to invariant subsets does not increase the size of the sector of analyticity in general.

2. AUXILIARY RESULTS

2.1. Functions convex in one direction. Let $\mathbf{F} = \{F_t\}_{t \geq 0} \subset \text{Hol}(\Pi)$ be a semigroup. It is known that in the case \mathbf{F} has no interior fixed points, Abel's functional equation

$$h(F_t(z)) = h(z) + t \quad (2.1)$$

has a unique up to a constant solution $h \in \text{Hol}(\Pi, \mathbb{C})$ which is a univalent function (see, for example, [15, 26]). The solution normalized by $h(1) = 0$ is called the associated Koenigs function of \mathbf{F} . Differentiating (2.1), we conclude that

$$h'(z)f(z) = 1. \quad (2.2)$$

In connection with equation (2.1), recall that a domain $D \subset \mathbb{C}$ is convex in the positive direction of the real axis, if $w + t \in D$ for all $w \in D$ and $t \geq 0$. The study of univalent mappings onto such domains was initiated by Robertson and continued by many mathematicians; see [20] and [15] for appropriate references and historical details.

Definition 2.1 (cf., [15, Definition 3.5]). *We say that a univalent function $h \in \text{Hol}(\Pi, \mathbb{C})$ normalized by $h(1) = 0$ is convex in direction $e^{i\theta}$, $\theta \in \mathbb{R}$, if for each $z \in \Pi$,*

$$h(z) + te^{i\theta} \in h(\Pi) \text{ for each } t > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} h^{-1}(h(z) + te^{i\theta}) = \infty. \quad (2.3)$$

We denote the class of all such functions by $\Sigma(\Pi, \theta)$ and let $\Sigma(\Pi) := \Sigma(\Pi, 0)$.

Remark 2.1. *It follows immediately from Definition 2.1 that $h \in \Sigma(\Pi, \theta)$ if and only if $\xi - te^{i\theta} \notin h(\Pi)$ for all $\xi \notin h(\Pi)$ and $t \geq 0$.*

In the sequel, we use the following characterization of functions univalent in Π and convex in the positive direction of the real axis.

Lemma 2.1. *A univalent function $h \in \text{Hol}(\Pi, \mathbb{C})$ is convex in the positive direction of the real axis if and only if $\text{Re } h'(z) \geq 0$.*

Proof. Let $C \in \text{Hol}(\mathbb{D}, \Pi)$ be the Cayley transform given by $C(w) = (1 + w)/(1 - w)$. The function $g = h \circ C$ is analytic in \mathbb{D} . Differentiating it, we obtain

$$g'(w) = h'(C(w))C'(w) = \frac{2h'(C(w))}{(1 - w)^2}.$$

By [15, Lemma 3.8], g is convex in the positive direction of the real axis if and only if $\operatorname{Re} g'(w)(1-w)^2 \geq 0$. Thus h is convex in the positive direction of the real axis if and only if $\operatorname{Re} h'(z) \geq 0$, $z \in \Pi$. \square

In what follows, we deal with functions which are convex in every direction in some sector. Namely, for $0 < \theta_1, \theta_2$ with $\theta_1 + \theta_2 \leq \pi$, we define

$$\Sigma(\Pi, \theta_1, \theta_2) = \bigcap_{\theta \in (-\theta_1, \theta_2)} \Sigma(\Pi, \theta) \quad (2.4)$$

and

$$\Omega(\theta_1, \theta_2) = \{\zeta : -\theta_1 < \arg \zeta < \theta_2\}. \quad (2.5)$$

Clearly, each function $h \in \Sigma(\Pi, \theta_1, \theta_2)$ satisfies

$$h(z) + \overline{\Omega(\theta_1, \theta_2)} \subset h(\Pi) \quad (2.6)$$

for all $z \in \Pi$. We complete this fact as follows (cf. [21]).

Proposition 2.1. *Let $h \in \Sigma(\Pi)$. Suppose that*

$$h(z) + \Omega(\theta_1, \theta_2) \subset h(\Pi) \quad \text{for all } z \in \Pi.$$

Then $h \in \Sigma(\Pi, \theta_1, \theta_2)$.

Proof. Fix $\theta \in (-\theta_1, \theta_2)$. Note that the formula

$$\Psi_t(z) = h^{-1}(h(z) + te^{i\theta})$$

defines a one-parameter continuous semigroup $\{\Psi_t\}_{t \geq 0} \subset \operatorname{Hol}(\Pi)$. Note that this semigroup has no interior fixed point. Indeed, if $z_0 \in \Pi$ is its fixed point $z_0 \in \Pi$, then

$$h(z_0) = h(z_0) + te^{i\theta}$$

for all $t > 0$, which is impossible. Hence, $\{\Psi_t\}_{t \geq 0} \subset \operatorname{Hol}(\Pi)$ has the boundary Denjoy–Wolff point ζ . We have to show that $\zeta = \infty$.

Let $a_0 \notin h(\Pi)$. By Remark 2.1, $a_0 - te^{i\beta} \notin h(\Pi)$ for all $\beta \in (-\theta_1, \theta_2)$ and then $a = a_0 - 1$ has a neighborhood contained in $\mathbb{C} \setminus h(\Pi)$. Thus the function $g \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ given by

$$g(w) = \frac{1}{h\left(\frac{1+w}{1-w}\right) - a} \quad (2.7)$$

is univalent and bounded in \mathbb{D} . Moreover, g satisfies

- (i) $0 \in \partial g(\mathbb{D})$;
- (ii) $\frac{g(w)}{1 - te^{i\theta}g(w)} \in g(\mathbb{D})$ whenever $t \geq 0$ and $\theta \in [-\theta_1, \theta_2]$;
- (iii) $\lim_{t \rightarrow \infty} g^{-1}\left(\frac{g(w)}{1 - tg(w)}\right) = 1$.

By the Carathéodory Theorem (see, for example, [8]), g induces the one-to-one correspondence between points of the unit circle $\partial\mathbb{D}$ and prime ends of $g(\mathbb{D})$. By property (iii), the impression $I(P(1))$ of the prime end $P(1)$ contains the point $\zeta_0 = 0$, which is its unique accessible point.

Let $\{C_n\}$ be a chain of crosscuts defining $P(1)$. For a fixed $w \in \mathbb{D}$, we can assume without loss of generality that each C_n separates $g(w)$ and ζ_0 . By property (ii),

$$G_w := \left\{ \frac{g(w)}{1 - te^{i\theta}g(w)} : t \geq 0, \theta \in [-\theta_1, \theta_2] \right\} \subset g(\mathbb{D}).$$

Thus, each crosscut C_n intersects every path joining $g(w)$ and ζ_0 that, with the exception of its end point ζ_0 , lies inside G_w . In particular, each C_n intersects the trajectory $\Gamma_w = \left\{ \frac{g(w)}{1 - te^{i\theta}g(w)} : t \geq 0 \right\}$, which tends to 0 as $t \rightarrow \infty$. So this trajectory tends to the same unique accessible point $\zeta_0 = 0$ of the prime end $P(1)$. Bearing in mind (2.7), we see that the last fact coincides with the assertion. \square

2.2. Semigroup generators in the right half-plane. Here we present some necessary and sufficient conditions for a function $f \in \text{Hol}(\Pi, \mathbb{C})$ to be a semigroup generator.

Proposition 2.2. *Let $f \in \text{Hol}(\Pi, \mathbb{C})$. Then f is a semigroup generator if and only if*

$$\text{Re} \left(f(z) \frac{\bar{z} - 1}{z + 1} \right) \leq \text{Re } z \cdot \text{Re} \left(f(1) \frac{\bar{z} - 1}{\bar{z} + 1} \right) \quad (2.8)$$

for all $z \in \Pi$.

Proof. To prove the necessity of (2.8), suppose that f generates a semigroup $\{F_t\}_{t \geq 0} \subset \text{Hol}(\Pi)$. Let $C \in \text{Hol}(\mathbb{D}, \Pi)$ be the Cayley transform of \mathbb{D} onto Π defined by $w \mapsto \frac{1+w}{1-w}$. It is easy to see that the family $\{\Phi_t\}_{t \geq 0}$ defined by $\Phi_t = C^{-1} \circ F_t \circ C$ forms a semigroup on \mathbb{D} . We find its generator g by differentiation:

$$g(w) = \lim_{t \rightarrow 0^+} \frac{\Phi_t(w) - w}{t} = (C^{-1})'(C(w))f(C(w)) = \frac{(1-w)^2 f(C(w))}{2}.$$

Since g is a semigroup generator in \mathbb{D} , it satisfies the flow-invariance condition $\text{Re}(g(w)\bar{w}) \leq (1 - |w|^2) \text{Re}(g(0)\bar{w})$ (see [26, Proposition 3.5.4] and [15,

Theorem 2.3]¹). Thus,

$$\operatorname{Re}((1-w)^2 f(C(w))\overline{w}) \leq (1-|w|^2) \operatorname{Re}(f(1)\overline{w}).$$

Substituting $w = C^{-1}(z) = \frac{z-1}{z+1}$, we obtain the required inequality.

Conversely, suppose that f satisfies (2.8).

Then the function $g \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ defined by $g(w) = \frac{1}{2}(1-w)^2 f(C(w))$ satisfies the flow-invariance condition, and hence generates a semigroup $\{\Phi_t\}_{t \geq 0}$ on \mathbb{D} . Then the functions $F_t = C \circ \Phi_t \circ C^{-1}$ form a semigroup on Π . Differentiating $\{F_t\}_{t \geq 0}$ with respect to t at 0, one sees that $\{F_t\}_{t \geq 0}$ is generated by f . \square

Proposition 2.3. *Let $f \in \operatorname{Hol}(\Pi, \mathbb{C})$ be a semigroup generator in Π . If $f'(\infty) := \angle \lim_{z \rightarrow \infty} \frac{f(z)}{z+1}$ exists and $f'(\infty) \geq 0$, then $\operatorname{Re} f(z) \geq 0$ for all $z \in \Pi$.*

Proof. Denote by $\mathbf{F} = \{F_t\}_{t \geq 0} \subset \operatorname{Hol}(\Pi)$ the semigroup generated by f and by $\Phi = \{\Phi_t\}_{t \geq 0} \subset \operatorname{Hol}(\mathbb{D})$ the semigroup defined by $\Phi_t = C^{-1} \circ F_t \circ C$. The generator $g \in \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ of Φ is given by $g(w) = \frac{1}{2}(1-w)^2 f(C(w))$. If $\angle \lim_{z \rightarrow \infty} \frac{f(z)}{z+1} = a \geq 0$, then

$$\angle \lim_{w \rightarrow 1} \frac{g(w)}{w-1} = \angle \lim_{w \rightarrow 1} \frac{(w-1)f(C(w))}{2} = \angle \lim_{z \rightarrow \infty} \frac{-f(z)}{z+1} = -a \leq 0.$$

Hence, by [15, Theorem 2.10], $w = 1$ is the Denjoy–Wolff point of Φ . In this case, \mathbf{F} has the Denjoy–Wolff point at ∞ . By Julia’s lemma, $\operatorname{Re} F_t(z)$ is a non-decreasing function in t . Now, by the definition of the semigroup generator, we conclude that $\operatorname{Re} f(z) \geq 0$ for all $z \in \Pi$. \square

3. SEMIGROUPS OF HOLOMORPHIC MAPPINGS

In this section, we find criteria for a semigroup of holomorphic self-mappings of the right half-plane Π to admit an analytic extension in its parameter to a given sector. It follows from these criteria that semigroups having repelling fixed points, as well semigroups of hyperbolic type, cannot be extended to any sector. Moreover, the same control functions that govern the size of the sector of analyticity enable us to localize the semigroup trajectories.

Theorem 3.1. *Let $f \in \operatorname{Hol}(\Pi, \mathbb{C})$ be a generator of the one-parameter continuous semigroup $\mathbf{F} = \{F_t\}_{t \geq 0} \subset \operatorname{Hol}(\Pi)$ with the Denjoy–Wolff point at ∞ , and let h be its associated Kœnigs function. For all $\theta_1, \theta_2 > 0$ such that $\theta_1 + \theta_2 \leq \pi$, the following are equivalent:*

¹In [15, Theorem 2.3], the opposite inequality sign appears. This is due to a slightly different definition.

- (i) \mathbf{F} can be analytically extended to a semigroup $\{F_\zeta\}_{\zeta \in \Omega(\theta_1, \theta_2)}$ such that for each $z \in \Pi$, the function $F_\zeta(z)$ tends to ∞ along every ray in $\Omega(\theta_1, \theta_2)$;
- (ii) the function f satisfies

$$-\frac{\pi}{2} + \theta_1 \leq \arg f(z) \leq \frac{\pi}{2} - \theta_2, \quad z \in \Pi; \quad (3.1)$$

- (iii) $h \in \Sigma(\Pi, \theta_1, \theta_2)$.

Proof. Step 1. Suppose that assertion (i) holds. Fix $z \in \Pi$. Since $\lim_{\mathbb{R} \ni \zeta \rightarrow 0} \frac{F_\zeta(z) - z}{\zeta} = f(z)$ and the semigroup \mathbf{F} is real-analytic in t in a neighborhood of $t = 0$, then by the uniqueness theorem, $\frac{F_\zeta(z) - z}{\zeta} \rightarrow f(z)$ as $\zeta \rightarrow 0$ along any path in \mathbb{C} . For $\theta \in (-\theta_1, \theta_2)$, define semigroup $\{\Psi_t\}_{t \geq 0}$ by $\Psi_t(z) = F_{te^{i\theta}}(z)$. Its generator ψ is given by

$$\psi(z) = \left. \frac{\partial \Psi_t(z)}{\partial t} \right|_{t=0} = \left. \frac{\partial F_{te^{i\theta}}(z)}{\partial t} \right|_{t=0} = \lim_{t \rightarrow 0^+} e^{i\theta} \frac{F_{te^{i\theta}}(z) - z}{te^{i\theta}} = e^{i\theta} f(z).$$

Since $\lim_{t \rightarrow \infty} F_{te^{i\theta}}(z) = \infty$, then $0 \leq \operatorname{Re} \psi(z) = \operatorname{Re} e^{i\theta} f(z)$. Hence, by the continuity of $e^{i\theta} f(z)$ in θ , we have $-\frac{\pi}{2} \leq \arg e^{i\theta} f(z) \leq \frac{\pi}{2}$ for all $\theta \in [-\theta_1, \theta_2]$, and (ii) follows.

Step 2. Assume now that (ii) holds. By formula (2.2), $\arg h'(z) = -\arg f(z)$. Then, by (3.1), we have $-\frac{\pi}{2} \leq \arg h'(z) - \theta \leq \frac{\pi}{2}$, from which it follows that $\operatorname{Re} e^{-i\theta} h'(z) \geq 0$ for all $\theta \in [-\theta_1, \theta_2]$.

By Lemma 2.1, function $e^{-i\theta} h$ is convex in the positive direction of the real axis, that is, $e^{-i\theta} h(z) + t \in e^{-i\theta} h(\Pi)$ for all $t \geq 0$. This implies that $h(z) + \zeta \in h(\Pi)$ for all $\zeta \in \Omega(\theta_1, \theta_2)$. Assertion (iii) now follows, by Proposition 2.1.

Step 3. Assertion (iii) implies that $F_\zeta(z) := h^{-1}(h(z) + \zeta)$ is well defined, analytic in $\zeta \in \Omega(\theta_1, \theta_2)$, and assumes values in Π for all $\zeta \in \Omega(\theta_1, \theta_2)$. For each pair $\zeta_1, \zeta_2 \in \Omega(\theta_1, \theta_2)$, we have $\zeta_1 + \zeta_2 \in \Omega(\theta_1, \theta_2)$ and

$$\begin{aligned} F_{\zeta_1} \circ F_{\zeta_2}(z) &= F_{\zeta_1}(\sigma^{-1}(\sigma(z) + \zeta_2)) = \\ &= \sigma^{-1}[\sigma(\sigma^{-1}(\sigma(z) + \zeta_2)) + \zeta_1] = \sigma^{-1}(\sigma(z) + \zeta_2 + \zeta_1) = F_{\zeta_1 + \zeta_2}(z). \end{aligned}$$

Finally, from Proposition 2.1, we conclude that

$$\lim_{t \rightarrow \infty} F_{te^{i\theta}} = \lim_{t \rightarrow \infty} h^{-1}(h(z) + te^{i\theta}) = \infty$$

for all $z \in \Pi$ and $\theta \in [-\theta_1, \theta_2]$, which establishes (i). \square

Example 3.1. Consider the affine semigroup $\mathbf{F} = \{F_t\}_{t \geq 0} \subset \text{Hol}(D)$, where $F_t(z) = z + At$ for some $A \in \Pi$. This semigroup is generated by the constant function $f(z) = A$. According to the equivalence of (ii) and (iii) of Theorem 3.1, \mathbf{F} has an analytic extension to the sector $\Omega(\frac{\pi}{2} + \arg A, \frac{\pi}{2} - \arg A)$. Indeed, let $F_\zeta(z) = z + A\zeta$. Then $\text{Re } F_\zeta(z) > 0$ for all $z \in \Pi$ whenever $\zeta \in \Omega(\frac{\pi}{2} + \arg A, \frac{\pi}{2} - \arg A)$. The same conclusion also follows from the equivalence of (iii) and (i).

The geometrical constructions used in Theorem 3.1 imply that if a semigroup \mathbf{F} has an analytical extension to a sector, then \mathbf{F} must be of parabolic type. Indeed, in the hyperbolic case, the planar domain $h(\Pi)$ is completely contained in a horizontal strip of finite width (see [9]), so condition (iii) of Theorem 3.1 is not satisfied.

By contrast, as in Example 3.1, there are parabolic type semigroups which admit an analytic extension to the half-plane $\{\zeta : |\arg \zeta| < \frac{\pi}{2}\}$. On the other hand, there exist parabolic type semigroups which cannot be extended. For example, each semigroup of automorphic type satisfies $d_{h(\Pi)}(w) = \lim_{s \rightarrow \infty} \delta_{h(\Pi)}(w + s) < \infty$. Hence its planar domain $h(\Pi)$ is contained in a horizontal half-plane. So no sector $\Omega \supset \mathbb{R}^+$ is contained in $h(\Pi)$, i.e., condition (iii) of Theorem 3.1 is not satisfied.

Moreover, the planar domain of a semigroup that has a regular repelling point contains a strip of the final maximal width; see [9] and [15]. Thus, condition (iii) of Theorem 3.1 cannot be satisfied, and the semigroup has no analytical extension to a sector.

Corollary 3.1. Let $\mathbf{F} = \{F_t\}_{t \geq 0} \subset \text{Hol}(\Pi)$ be a one-parameter continuous semigroup with the Denjoy–Wolff point at ∞ . Suppose that \mathbf{F} can be extended to a semigroup $\{F_\zeta\}_{\zeta \in \Omega}$ which is analytic in some sector Ω and such that $F_\zeta(z) \rightarrow \infty$ along every ray in Ω for each $z \in \Pi$. Then \mathbf{F} is of parabolic type and has no boundary regular repelling points.

Proof. Note that \mathbf{F} has no interior fixed point since it has the Denjoy–Wolff point at ∞ . Denote the generator of \mathbf{F} by $f \in \text{Hol}(\Pi)$. In the hyperbolic case, $\text{Re } f(z) \geq 0$ and $\angle \lim_{z \rightarrow \infty} \frac{f(z)}{z} = a > 0$. Hence, for every $\theta \neq 0$, the function $e^{i\theta} f(z)$ is not a semigroup generator, and condition (ii) fails. Therefore, \mathbf{F} must be of parabolic type.

Suppose by contrary, that \mathbf{F} has a boundary repelling fixed point $ib \in \partial\Pi$, $b \in \mathbb{R}$. Then $f(z) = (z - ib)(f'(ib) + \rho(z))$, where $f'(ib) > 0$ and $\angle \lim_{z \rightarrow ib} \rho(z) = 0$. Setting $z_r = ib + re^{i\theta}$ for a fixed $\theta \in (-\pi/2, \pi/2)$, we see that for small enough r , $\arg f(z_r) = \theta + \arg(f'(ib) + \rho(z_r))$ is close to θ . By Theorem 3.1 (ii), this contradicts the analytic extendibility of \mathbf{F} . \square

Note that the parabolic group $\{F_t\}_{t \in \mathbb{R}}$, $F_t(z) = z + iqt$ with $q \in \mathbb{R}$ admits an analytic extension to the horizontal half-plane $\{\zeta : q \operatorname{Im} \zeta \leq 0\}$.

Corollary 3.1 implies the following fact.

Corollary 3.2. *Let Ω be a sector which is not contained in a horizontal half-plane. There is no group $\mathbf{F} = \{F_t\}_{t \in \mathbb{R}} \subset \operatorname{Hol}(\Pi)$ that can be analytically extended in its parameter to Ω .*

Proof. If such a group exists, it cannot be of elliptic type. Hence it has the Denjoy–Wolff point $\tau \in \partial\Pi \cup \{\infty\}$. If $\tau \neq \infty$, then the group $\{G_t\}_{t \in \mathbb{R}}$ defined by

$$G_t(z) = \frac{1}{F_t\left(\frac{1}{z} + \tau\right) - \tau}$$

has the Denjoy–Wolff point $\tilde{\tau} = \infty$. By Corollary 3.1, $\{G_t\}$ is of parabolic type, that is, $G_t(z) = z + iqt$ for some $q \in \mathbb{R}$. Clearly, there is no sector Ω such that $G_\zeta(\Pi) \subset \Pi$ for all $\zeta \in \Omega$. \square

We now turn to semigroup generators whose arguments are controlled by functions given a priori.

Theorem 3.2. *Let $\gamma_1, \gamma_2 : \mathbb{R}^+ \rightarrow (0, \frac{\pi}{2})$ be decreasing functions. Let $\mathbf{F} = \{F_t\}_{t \geq 0} \subset \operatorname{Hol}(\Pi)$ be a semigroup generated by $f \in \operatorname{Hol}(\Pi, \mathbb{C})$ which satisfies*

$$-\gamma_1(\operatorname{Re} z) \leq \arg f(z) \leq \gamma_2(\operatorname{Re} z) \quad \text{for all } z \in \Pi. \quad (3.2)$$

Then for each $k > 0$, the semigroup $\mathbf{F}^k = \{\Phi_t\}_{t \geq 0}$ defined by $\Phi_t(z) = F_t(z + k) - k$ can be analytically extended in t to the sector $\Omega\left(\frac{\pi}{2} - \gamma_1(k), \frac{\pi}{2} - \gamma_2(k)\right)$. In particular, if $\lim_{s \rightarrow \infty} \gamma_1(s) = \lim_{s \rightarrow \infty} \gamma_2(s) = 0$, then for each $\varepsilon > 0$, there exists $k > 0$ such that \mathbf{F}^k has an analytic extension to the sector $\Omega\left(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} - \varepsilon\right)$.

The result is a direct consequence of the equivalence (ii) \Leftrightarrow (i) of Theorem 3.1.

In fact, the semigroups \mathbf{F}^k that appear in Theorem 3.2 are the restrictions of the original semigroup \mathbf{F} to sub-half-planes of Π , so Theorem 3.2 gives a criterion for the analytic extendibility of a semigroup considered not on the whole domain but on its invariant subsets. It follows from [16] that the restriction of a semigroup on \mathbb{D} with the Denjoy–Wolff point $\tau = 0$ to an invariant subset (which is a disk of radius $r < 1$) can be analytically extended to a sector, and the sector is wider the smaller r is. In our case, the situation is different. As the following example shows, in general, the restriction of a semigroup to invariant subsets need not increase the size of the sector to which the semigroup can be extended.

Example 3.2. Fix $\alpha \in (0, 1)$, and consider the semigroup generator defined by $f(z) = z^{1-\alpha}$. Obviously, $|\arg f(z)| \leq \frac{\pi(1-\alpha)}{2}$, for all $z \in \Pi$; hence, by Theorems 3.1–3.2, the generated semigroup $\mathbf{F} = \{F_t\}_{t \geq 0}$ can be extended to the sector $\Omega\left(\frac{\pi\alpha}{2}, \frac{\pi\alpha}{2}\right)$. This semigroup can be found directly as the unique solution of the Cauchy problem

$$\begin{cases} \frac{\partial F_t(z)}{\partial t} = (F_t(z))^{1-\alpha}, \\ F_0(z) = z. \end{cases}$$

So, $F_t(z) = (\alpha t + z^\alpha)^{1/\alpha}$. For each $k > 0$, the restriction of \mathbf{F} to the half-plane $\{z : \operatorname{Re} z > k\}$, or, more precisely, the semigroup $\mathbf{F}^k = \{\Phi_t\}_{t \geq 0} \in \operatorname{Hol}(\Pi)$ defined by $\Phi_t(z) = F_t(z+k) - k$, is generated by $f^k(z) = (z+k)^{1-\alpha}$. Therefore, \mathbf{F}^k can be analytically extended to the same sector $\Omega\left(\frac{\pi\alpha}{2}, \frac{\pi\alpha}{2}\right)$ but to no larger sector.

It turns out that the same control functions which prescribe in Theorem 3.2 the size of the sector of analyticity for a semigroup provide a localization of semigroup trajectories in the sense of differential inclusions.

Proposition 3.1. Let $\{F_t\}_{t \geq 0} \in \operatorname{Hol}(\Pi)$ be a semigroup generated by $f \in \operatorname{Hol}(\Pi, \mathbb{C})$. Suppose that there exist continuous functions $\gamma_1, \gamma_2 : \mathbb{R}^+ \rightarrow (0, \frac{\pi}{2})$, such that

$$-\gamma_1(\operatorname{Re} z) \leq \arg f(z) \leq \gamma_2(\operatorname{Re} z) \text{ for all } z \in \Pi. \quad (3.3)$$

Then, for each initial point $z \in \Pi$, the trajectory $\{F_t(z), t \geq 0\}$ lies in the domain

$$\{u + iv : u \geq \operatorname{Re} z, B_1(u) \leq v \leq B_2(u)\}, \quad (3.4)$$

where

$$\begin{aligned} B_1(u) &= \operatorname{Im} z - \int_{\operatorname{Re} z}^u \tan \gamma_1(s) ds, \\ B_2(u) &= \operatorname{Im} z + \int_{\operatorname{Re} z}^u \tan \gamma_2(s) ds. \end{aligned} \quad (3.5)$$

Proof. Fix $z \in \Pi$. Separating the real and imaginary parts, we write

$$F_t(z) = u_t(z) + iv_t(z) \quad \text{and} \quad f(z) = \phi(z) + i\psi(z).$$

Then $u_0 = \operatorname{Re} z$, $v_0 = \operatorname{Im} z$, $\frac{\partial u_t}{\partial t} = \phi(u_t + iv_t) > 0$, and $\frac{\partial v_t}{\partial t} = \psi(u_t + iv_t)$. Therefore, along the semigroup trajectory $\{F_t(z) : t \geq 0\}$, the function u_t is strictly increasing. Hence, v_t can be considered as a function of u_t . Furthermore, $\frac{dv_t}{du_t} = \tan(\arg f(u_t + iv_t))$. Consequently, by (3.3),

$$-\tan \gamma_1(u_t) \leq \frac{dv_t}{du_t} \leq \tan \gamma_2(u_t).$$

Integrating this inequality with respect to u_t , we obtain

$$-\int_{u_0}^{u_t} \tan \gamma_1(s) ds \leq v_t(u_t) - v_0 \leq \int_{u_0}^{u_t} \tan \gamma_2(s) ds$$

for all $t \geq 0$. The proof is complete. \square

Example 3.3. Let $f \in \text{Hol}(\Pi)$ be defined by $f(z) = \frac{z+a}{z+b}$, $0 \leq a < b$. Since

$\lim_{z \rightarrow \infty} \frac{f(z)}{z+1} = 0$, f generates a semigroup of parabolic type which is the unique solution of the Cauchy problem

$$\begin{cases} \frac{\partial F_t(z)}{\partial t} = \frac{F_t(z) + a}{F_t(z) + b} \\ F_0(z) = z, z \in \Pi, \end{cases}$$

and hence satisfies the functional equation

$$F_t(z) + (b-a) \log(F_t(z) + a) = t + z + (b-a) \log(z + a).$$

This equation, being transcendental, does not allow localization of the semigroup trajectories. Nevertheless, it is easy to see that

$$|\arg f(z)| \leq \arctan \frac{b-a}{2\sqrt{(\operatorname{Re} z + a)(\operatorname{Re} z + b)}} =: \gamma(\operatorname{Re} z).$$

Therefore, by Proposition 3.1,

$$|\operatorname{Im} F_t(z) - \operatorname{Im} z| \leq \frac{b-a}{2} \log \frac{\frac{a+b}{2} + \operatorname{Re} F_t(z) + \sqrt{(\operatorname{Re} F_t(z) + a)(\operatorname{Re} F_t(z) + b)}}{\frac{a+b}{2} + \operatorname{Re} z + \sqrt{(\operatorname{Re} z + a)(\operatorname{Re} z + b)}}$$

for all $z \in \Pi$ and $t \geq 0$.

Furthermore, according to Theorems 3.1 and 3.2, the semigroup \mathbf{F} restricted to the half-plane $\{z : \operatorname{Re} z > k\}$ can be analytically extended to the sector $\Omega\left(\frac{\pi}{2} - \gamma(k), \frac{\pi}{2} - \gamma(k)\right)$. A direct calculation shows that for all $\varepsilon > 0$ and $k > b-a - (a+b) \sin \varepsilon$, the restricted semigroup \mathbf{F}^k has an analytic extension to the sector $\Omega\left(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} - \varepsilon\right)$.

Theorem 3.2 can also be used to study the important classes $\mathcal{G}_\alpha(\Pi)$, $\alpha > 0$, of semigroup generators of the form

$$f(z) = A(z+1)^{1-\alpha} + \varrho(z), \quad \text{where} \quad \lim_{z \rightarrow \infty} \frac{\varrho(z)}{(z+1)^{1-\alpha}} = 0 \quad (3.6)$$

and $A > 0$. Note that for $f \in \mathcal{G}_\alpha(\Pi)$,

$$\lim_{\operatorname{Re} z \rightarrow \infty} \arg \frac{f(z)}{A(z+1)^{1-\alpha}} = 0. \quad (3.7)$$

These classes were first introduced in [17] for semigroup generators in the open unit disk and then studied in more detail in [13, 14]. In particular, it was shown that if $f \in \mathcal{G}_\alpha(\Pi)$, then $\alpha \leq 2$ and $|\arg A| \leq \frac{\pi}{2} \min\{\alpha, 2 - \alpha\}$. For $0 < \alpha < 2$, we now consider the slightly narrower class of generators for which $|\arg A| < \frac{\pi}{2} \min\{\alpha, 2 - \alpha\}$. The functions δ_1, δ_2 defined on $(0, \infty)$ by

$$\begin{aligned}\delta_1(s) &= - \inf_{\operatorname{Re} z \geq s} \arg \frac{f(z)}{A(z+1)^{1-\alpha}}, \\ \delta_2(s) &= \sup_{\operatorname{Re} z \geq s} \arg \frac{f(z)}{A(z+1)^{1-\alpha}}\end{aligned}\tag{3.8}$$

are non-increasing and, by (3.7), tend to 0 as $s \rightarrow \infty$. Moreover, for every $z \in \Pi$,

$$\begin{aligned}(1 - \alpha) \arg(1 + z) + \arg A - \delta_1(\operatorname{Re} z) &\leq \arg f(z) \\ &\leq (1 - \alpha) \arg(1 + z) + \arg A + \delta_2(\operatorname{Re} z),\end{aligned}$$

that is, inequality (3.2) holds with

$$\begin{aligned}\gamma_1(s) &= \frac{|1 - \alpha|\pi}{2} - \arg A + \delta_1(s), \\ \gamma_2(s) &= \frac{|1 - \alpha|\pi}{2} + \arg A + \delta_2(s).\end{aligned}$$

This establishes the following consequence of Theorem 3.2.

Corollary 3.3. *Let $\{F_t\}_{t \geq 0} \in \operatorname{Hol}(\Pi)$ be a semigroup generated by $f \in \mathcal{G}_\alpha(\Pi)$, $\alpha \in (0, 2)$, such that (3.6) holds with $|\arg A| < \frac{\pi}{2} \min\{\alpha, 2 - \alpha\}$. Then for each $k > 0$, the semigroup $\mathbf{F}^k = \{\Phi_t\}_{t \geq 0} \in \operatorname{Hol}(\Pi)$ defined by $\Phi_t(z) = F_t(z+k) - k$ can be analytically extended to the sector $\Omega(\theta_1(k), \theta_2(k))$, where*

$$\begin{aligned}\theta_1(k) &= \frac{\pi}{2} \min\{\alpha, 2 - \alpha\} + \arg A - \delta_1(k), \\ \theta_2(k) &= \frac{\pi}{2} \min\{\alpha, 2 - \alpha\} - \arg A - \delta_2(k),\end{aligned}$$

and δ_1, δ_2 are defined by (3.8). Consequently, for each sector $\tilde{\Omega}$ such that

$$\tilde{\Omega} \setminus \{0\} \subset \Omega\left(\frac{\pi}{2} \min\{\alpha, 2 - \alpha\} + \arg A, \frac{\pi}{2} \min\{\alpha, 2 - \alpha\} - \arg A\right),$$

there exists $k > 0$ such that \mathbf{F}^k has an analytic extension to $\tilde{\Omega}$.

Remark 3.1. *Note that the union $\bigcup_{\alpha \in (0, 2]} \mathcal{G}_\alpha(\Pi)$ provides a wide class of semigroup generators for which the so-called Slope-problem has affirmative answer. This problem that was posed in [9] and stayed open for about ten years, can be formulated as follows. For $z \in \Pi$, is the set $\operatorname{Slope}^+(F_t(z))$ of*

all limit points of the curve $\{(t, \arg F_t(z)) : t \in (0, \infty)\}$ as $t \rightarrow \infty$ always a singleton? This question was answered negatively in general quite recently in [6] and [10].

Let $\{F_t\}_{t>0}$ and f satisfy the hypotheses of Proposition 3.1, and let $z \in \Pi$. Assume that $\limsup_{u \rightarrow \infty} \frac{B_1(u)}{u} > \liminf_{u \rightarrow \infty} \frac{B_2(u)}{u}$, where B_1 and B_2 are defined by (3.5). Proposition 3.1 implies that $\text{Slope}^+(F_t(z))$ is a segment. Moreover, the parabolic type semigroup generated by f_n , where $f_n(z) = f(z)^{\frac{1}{n+1}}$, has an analytic extension to a sector with angle close to π , and $\text{Slope}^+(F_t(z))$ is not a singleton.

4. SEMIGROUPS OF COMPOSITION OPERATORS

In this section, for a given semigroup $\mathbf{F} = \{F_t\}_{t \geq 0} \subset \text{Hol}(\Pi)$, we study the semigroup of composition operators $C_t : \phi \mapsto \phi \circ F_t$ on Hardy spaces $H^p(\Pi)$, $p \geq 1$, and solve the problem of its analytic extension.

It was shown in [2, Theorem 3.3] that if f generates a semigroup $\mathbf{F} \subset \text{Hol}(\Pi)$ which induces a semigroup \mathbf{T} of bounded composition operators on $H^p(\Pi)$, then \mathbf{T} is generated by the operator $\Gamma : \varphi \mapsto \varphi' f$. From the point of view of the Lumer–Phillips theorem, our first result is, in a sense, a converse assertion.

Theorem 4.1. *Let $f \in \text{Hol}(\Pi, \mathbb{C})$ be such that $\frac{f(z)}{z+1}$ is bounded in $\{z : \text{Re } z > 0, |z| > R\}$ for some $R > 0$. Fix $p \in (1, \infty)$ and define the linear operator $\Gamma := \Gamma_p$ on the domain $\mathcal{D}_p := \{\varphi \in H^p(\Pi) : \varphi' \cdot f \in H^p(\Pi)\}$ by $\Gamma\varphi(z) = \varphi'(z)f(z)$. If Γ is dissipative, then f is a semigroup generator in Π . If, in addition, $I - \Gamma$ is surjective, then f generates a semigroup with the Denjoy–Wolff point at ∞ .*

Proof. Recall that the dissipativity of the operator Γ means that for each $\varphi \in \mathcal{D}_p$, $\|\varphi\| = 1$, there exists a support functional φ^* such that $\text{Re}\langle \Gamma\varphi, \varphi^* \rangle \leq 0$. Note that in the space $H^p(\Pi)$, $p > 1$, the support functional is unique and can be realized as follows:

$$\varphi^*(\psi) = \langle \psi, \varphi^* \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi(iy) \frac{|\varphi(iy)|^p}{\varphi(iy)} dy. \quad (4.1)$$

For $a \in \Pi \setminus \{1\}$ and $n \in \mathbb{N}$, set $\varphi_n(z) = \frac{(z-a)^n}{(z+1)^{\frac{2}{p}}(z+\bar{a})^n}$. First we show that $\varphi_n \in \mathcal{D}_p$ and $\|\varphi_n\| = 1$ for each n . Indeed,

$$|\varphi_n(z)|^p = \frac{1}{|z+1|^2} \left| \frac{z-a}{z+\bar{a}} \right|^n \leq \frac{1}{|z+1|^2}, \quad |\varphi_n(iy)|^p = \frac{1}{y^2+1}.$$

Therefore $\varphi_n \in H^p(\Pi)$ and $\|\varphi_n\|^p = \frac{1}{\pi} \int_{-\infty}^{\infty} |\varphi_n(iy)|^p dy = 1$. In addition,

$$\begin{aligned} \varphi'_n(z)f(z) &= \left(f(z) \frac{\varphi'_n(z)}{\varphi_n(z)} \right) \varphi_n(z) \\ &= f(z) \left(\frac{n}{z-a} - \frac{2}{p(z+1)} - \frac{n}{z+\bar{a}} \right) \varphi_n(z). \end{aligned}$$

The boundedness of $\frac{f(z)}{z+1}$ implies that $\varphi'_n f \in H^p(\Pi)$, so $\varphi_n \in \mathcal{D}_p$.

We now calculate $\langle \Gamma \varphi_n, \varphi_n^* \rangle$ directly. Using (4.1), we have

$$\begin{aligned} \langle \Gamma \varphi_n, \varphi_n^* \rangle &= \langle \varphi'_n f, \varphi_n^* \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} f(iy) \frac{\varphi'_n(iy)}{\varphi_n(iy)} |\varphi_n(iy)|^p dy \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(iy) \left(\frac{n}{iy-a} - \frac{2}{p(iy+1)} - \frac{n}{iy+\bar{a}} \right) \frac{dy}{y^2+1} \end{aligned}$$

Applying (for instance) the inverse Cayley transform and the Cauchy formula, we obtain

$$\begin{aligned} \langle \Gamma \varphi_n, \varphi_n^* \rangle &= f(1) \left(-\frac{1}{p} - \frac{n}{1+\bar{a}} - \frac{n}{a-1} \right) + f(a) \frac{2n}{a^2-1} \\ &= \frac{2n}{|a-1|^2} \left[f(a) \frac{\bar{a}-1}{a+1} - f(1) \left(\operatorname{Re} a \cdot \frac{\bar{a}-1}{\bar{a}+1} \right) \right] - \frac{f(1)}{p}. \end{aligned}$$

Since $\operatorname{Re} \langle \Gamma \varphi_n, \varphi_n^* \rangle \leq 0$ for arbitrarily large n , it follows that

$$\operatorname{Re} \left[f(a) \frac{\bar{a}-1}{a+1} - f(1) \operatorname{Re} a \cdot \frac{\bar{a}-1}{\bar{a}+1} \right] \leq 0.$$

Thus, by Proposition 2.2, f is a semigroup generator.

If $I - \Gamma$ is surjective, then by the Lumer–Phillips theorem, Γ generates a semigroup $\mathbf{T} = \{T(t), t \geq 0\}$ with $\|T(t)\| \leq 1$ for all $t \geq 0$. On the other hand, by [2, Theorem 2.8], $\|T(t)\| = e^{-\delta t/p}$, where δ is the angular derivative of f at ∞ . Therefore, $\delta \geq 0$. Hence, f generates a semigroup with the Denjoy–Wolff point at ∞ . \square

Next we establish a characterization of composition operators on the Hardy spaces (cf. [4, Theorem 2.5]). For $n = 0, 1, 2, \dots$, define $e_n \in H^p(\Pi)$ by

$$e_n(z) = \frac{(z-1)^n}{(z+1)^{\frac{2}{p}+n}}.$$

Theorem 4.2. *Let $1 \leq p < \infty$, and let T be a bounded linear operator on the Hardy space $H^p(\Pi)$. Then T is a composition operator if and only if all of the following three conditions hold:*

- (a) the function $F := \frac{1}{(Te_0)^{\frac{p}{2}}} - 1$ is well defined and belongs to $\text{Hol}(\Pi)$;
- (b) the angular derivative $\angle \lim_{z \rightarrow \infty} \frac{F(z)}{z}$ exists and is different from zero;
- (c) the operator T satisfies $Te_n = Te_0 \cdot \left(1 - 2(Te_0)^{\frac{p}{2}}\right)^n$ for all $n \in \mathbb{N}$.

Moreover, if these conditions hold, then $T\phi = \phi \circ F$ for all $\phi \in H^p(\Pi)$.

Proof. Suppose that T is a composition operator, say, $T\phi = \phi \circ G$ for some $G \in \text{Hol}(\Pi)$. Then $F = \frac{1}{(e_0 \circ G)^{\frac{p}{2}}} - 1 = G$, so condition (a) holds. Since T is bounded, condition (b) is also satisfied (see [19]). The verification of condition (c) is direct.

Conversely, suppose that conditions (a), (b) and (c) hold. We wish to show that $T\phi = \phi \circ F$. Indeed, $e_0 \circ F = Te_0$, and

$$\begin{aligned} e_n \circ F &= \frac{(F(z) - 1)^n}{(F(z) + 1)^{\frac{2}{p} + n}} = \frac{\left(\frac{1}{(Te_0)^{\frac{p}{2}}} - 2\right)^n}{\left(\frac{1}{Te_0}\right)^{\frac{p}{2}\left(\frac{2}{p} + n\right)}} \\ &= (Te_0)^{1 + \frac{pn}{2}} \left(\frac{1}{(Te_0)^{\frac{p}{2}}} - 2\right)^n = Te_0 \left(1 - 2(Te_0)^{\frac{p}{2}}\right)^n = Te_n \end{aligned}$$

for $n = 1, 2, 3, \dots$. Thus, the linearity and the continuity of both T and the operator of composition with F imply that $T\phi = \phi \circ F$ for all $\phi \in H^p(\Pi)$. \square

Now we are ready to present the main result of this section.

Theorem 4.3. *Let $f \in \text{Hol}(\Pi, \mathbb{C})$ be such that $\text{Re } f(z) > 0$ for all $z \in \Pi$, and $\lim_{z \rightarrow \infty} \frac{f(z)}{z + 1} = 0$. Fix $p \in (1, \infty)$ and define the linear operator $\Gamma := \Gamma_p$ on the domain $\mathcal{D}_p := \{\varphi \in H^p(\Pi) : \varphi' \cdot f \in H^p(\Pi)\}$ by $\Gamma\varphi(z) = \varphi'(z)f(z)$. For all $\theta_1, \theta_2 > 0$ such that $\theta_1 + \theta_2 \leq \pi$, the following assertions are equivalent:*

- (i) the function f satisfies inequality (3.1), that is,

$$-\frac{\pi}{2} + \theta_1 \leq \arg f(z) \leq \frac{\pi}{2} - \theta_2, \quad z \in \Pi;$$

- (ii) the operators $e^{-i\theta_1}\Gamma$ and $e^{i\theta_2}\Gamma$ generate semigroups of contractions on the space $H^p(\Pi)$;
- (iii) the semigroup $\{T(t), t \geq 0\}$ generated by Γ can be analytically extended to the semigroup $\{T(\zeta), \zeta \in \Omega(\theta_1, \theta_2)\}$ consisting of composition operators with $\|T(\zeta)\| = 1$ on $H^p(\Pi)$ for all $\zeta \in \Omega(\theta_1, \theta_2)$.

Proof. First we note that the operator $\Gamma^\theta := e^{i\theta}\Gamma$ has the same domain \mathcal{D}_p for every $\theta \in \mathbb{R}$.

Step 1. Suppose that assertion (i) holds. For $\theta \in [-\theta_1, \theta_2]$, let $f_\theta(z) = e^{i\theta}f(z)$. By assumption, $\operatorname{Re} f_\theta(z) \geq 0$ and $\lim_{z \rightarrow \infty} \frac{f_\theta(z)}{z+1} = 0$. Therefore, f_θ generates a one-parameter parabolic type semigroup $\{F_t^\theta\}_{t \geq 0} \subset \operatorname{Hol}(\Pi)$ with the Denjoy–Wolff point at ∞ . Thus, the composition operators C_t^θ defined by $C_t^\theta \phi := \phi \circ F_t^\theta$ are bounded on $H^p(\Pi)$, and $\|C_t^\theta\| = 1$ for all $t \geq 0$ (see [2, Theorem 2.8] and [19, Corollary 3.5]). By [2, Theorem 3.3], the infinitesimal generator of the contractive semigroup $\{C_t^\theta\}$ is Γ^θ . So, assertion (ii) follows (notice that the implication (i) \Rightarrow (iii) is proved too).

Step 2. Assume now that (ii) holds. By the Lumer–Phillips theorem, the operators $\Gamma^{-\theta_1} = e^{-i\theta_1}\Gamma$ and $\Gamma^{\theta_2} = e^{i\theta_2}\Gamma$ are dissipative; and for each $\lambda > 0$, the operators $\lambda I - \Gamma^{-\theta_1}$ and $\lambda I - \Gamma^{\theta_2}$ are surjective. Thus both the resolvent sets of $\lambda I - \Gamma^{-\theta_1}$ and of $\lambda I - \Gamma^{\theta_2}$ contain the right half-plane (see Remark 5.8 in [23]). Therefore, for every $\theta \in [-\frac{\pi}{2} - \theta_1, \theta_2 + \frac{\pi}{2}]$, the operator $I - e^{i\theta}\Gamma$ is surjective. Moreover,

$$\operatorname{Re} e^{-i\theta_1} \langle \Gamma \varphi, \varphi^* \rangle = \operatorname{Re} \langle e^{-i\theta_1} \Gamma \varphi, \varphi^* \rangle \leq 0$$

for some support functional φ^* at the point $\varphi \in H^p(\Pi)$, which yields

$$\frac{\pi}{2} + \theta_1 \leq \arg \langle \Gamma \varphi, \varphi^* \rangle \leq \frac{3\pi}{2} + \theta_1.$$

Similarly,

$$\frac{\pi}{2} - \theta_2 \leq \arg \langle \Gamma \varphi, \varphi^* \rangle \leq \frac{3\pi}{2} - \theta_2.$$

Therefore,

$$\frac{\pi}{2} + \theta_1 \leq \arg \langle \Gamma \varphi, \varphi^* \rangle \leq \frac{3\pi}{2} - \theta_2,$$

which means that

$$\operatorname{Re} \langle e^{i\theta} \Gamma \varphi, \varphi^* \rangle \leq 0 \tag{4.2}$$

for all $\theta \in [-\theta_1, \theta_2]$.

Consequently, for each $\theta \in [-\theta_1, \theta_2]$, the operator $\Gamma^\theta = e^{i\theta}\Gamma$ is dissipative, and $I - \Gamma^\theta$ is surjective. The Lumer–Phillips theorem then implies that each Γ^θ generates a one-parameter semigroup of contractions. Moreover, by [1, Theorem 2.4], the contractive semigroup $\{T(t), t \geq 0\}$ generated by Γ can be extended to the semigroup $\{T(\zeta)\}$ analytic in $\zeta \in \Omega(\theta_1, \theta_2)$.

Note also that by our assumptions, f generates a parabolic type semigroup $\mathbf{F} = \{F_t\}_{t \geq 0}$ which, in turn, induces the semigroup of composition operators $\mathbf{T} = \{T(t), t \geq 0\}$ defined by $T(t)\varphi = \varphi \circ F_t$ and generated by Γ . For each

$n \in \mathbb{N}$, the function $\ell_n : \Omega(\theta_1, \theta_2) \mapsto H^p(\Pi)$ defined by

$$\ell_n(\zeta) = T(\zeta)e_n - T(\zeta)e_0 \left(1 - 2(T(\zeta)e_0)^{\frac{p}{2}}\right)^n$$

is analytic in $\Omega(\theta_1, \theta_2)$. Since $\{T(t), t \geq 0\}$ is a composition semigroup, each ℓ_n vanishes on \mathbb{R}^+ by Theorem 4.2. By the uniqueness theorem, $\ell_n \equiv 0$ in $\Omega(\theta_1, \theta_2)$. Hence, by Theorem 4.2, $T(\zeta)$ is a composition operator for each $\zeta \in \Omega(\theta_1, \theta_2)$. This proves (ii) \Rightarrow (iii). (Note in passing that the implication (iii) \Rightarrow (ii) is obvious.)

Step 3. Assertion (iii) implies that there exists a semigroup $\tilde{\mathbf{F}} = \{\tilde{F}_t\}_{t \geq 0} \subset \text{Hol}(\Pi)$ that induces a semigroup of composition operators $\{\tilde{C}_t\}$ generated by $\Gamma^{-\theta_1} := e^{-i\theta_1}\Gamma$. The inequality

$$\left| \phi \circ \tilde{F}_t(z) - \phi(z) \right| \leq \frac{\|\tilde{C}_t\phi - \phi\|}{(4\pi \operatorname{Re} z)^{1/p}}$$

(cf. Lemma 3.2 in [2]) implies that $\tilde{\mathbf{F}}$ is a continuous (hence, generated) semigroup on $\text{Hol}(\Pi)$. Denote its generator by \tilde{f} .

Since the norm of composition operators \tilde{C}_t equals 1 for all t , these operators are bounded on $H^p(\Pi)$. By [2, Theorem 3.3], the infinitesimal generator of $\{\tilde{C}_t\}$ is $\Gamma^{-\theta_1}\varphi = \varphi'\tilde{f}$. On the other hand, $\Gamma^{-\theta_1}\varphi = e^{-i\theta_1}\Gamma\varphi = e^{-i\theta_1}\varphi'f$, so $\tilde{f}(z) = e^{-i\theta_1}f(z)$. Thus, $\lim_{z \rightarrow \infty} \frac{\tilde{f}(z)}{z+1} = 0$, that is, $\tilde{\mathbf{F}}$ is of parabolic type with the Denjoy–Wolff point at ∞ . Therefore, $\operatorname{Re} \tilde{f}(z) = \operatorname{Re} e^{-i\theta_1}f(z) \geq 0$. Similarly, $\operatorname{Re} e^{i\theta_2}f(z) \geq 0$. These last two inequalities imply (i). \square

In fact, the equivalence of assertions (ii) and (iii) to assertion (i) implies the following fact.

Corollary 4.1. *If either assertion (ii) or (iii) of Theorem 4.3 holds for some $p > 1$, then both assertions hold for every $p > 1$.*

Remark 4.1. *We have already seen in the proof of Theorem 4.3 that if (ii) holds, then the operator $e^{i\theta}\Gamma$ is dissipative for every $\theta \in [-\theta_1, \theta_2]$. Thus, by Theorem 4.1, each function $e^{i\theta}f$ is a semigroup generator. In addition, by the assumptions of Theorem 4.3, $\lim_{z \rightarrow \infty} \frac{e^{i\theta}f(z)}{z+1} = 0$. Therefore, by Proposition 2.3, $\operatorname{Re} e^{i\theta}f(z) \geq 0$ for all $z \in \Pi$. This shows independently that (i) follows from (ii).*

The next result follows from Theorems 3.2 and 4.3.

Corollary 4.2. *Let $\gamma_1, \gamma_2 : \mathbb{R}^+ \rightarrow (0, \frac{\pi}{2})$ be decreasing functions. Assume that $f \in \text{Hol}(\Pi, \mathbb{C})$ satisfies*

$$-\gamma_1(\text{Re } z) \leq \arg f(z) \leq \gamma_2(\text{Re } z) \quad \text{for all } z \in \Pi.$$

Let $1 < p < \infty$. For each $k > 0$, define the linear operator Γ^k on the domain $\mathcal{D}_{p,k} := \{\varphi \in H^p(\Pi) : \varphi' f(\cdot + k) \in H^p(\Pi)\}$ by $\Gamma^k \varphi(z) = \varphi'(z) f(z + k)$. Then Γ^k generates a semigroup of composition operators which can be analytically extended to the sector $\Omega(\frac{\pi}{2} - \gamma_1(k), \frac{\pi}{2} - \gamma_2(k))$. In particular, if $\lim_{s \rightarrow \infty} \gamma_1(s) = \lim_{s \rightarrow \infty} \gamma_2(s) = 0$, then for each $\varepsilon > 0$, there exists a $k > 0$ such that the composition semigroup generated by Γ^k is analytic in the sector $\Omega(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} - \varepsilon)$.

Example 4.1. *Let us return to the function $f(z) = \frac{z+a}{z+b}$, $0 < a < b$, considered in Example 3.3. For $k > 0$, consider now the linear operator Γ^k , $k \geq 0$, defined by $\Gamma^k : \varphi(z) \mapsto \varphi'(z) f(z + k) = \varphi'(z) \frac{z+k+a}{z+k+b}$. By Corollary 4.2, Γ_k generates a semigroup of composition operators which can be analytically extended to the sector $\Omega(\frac{\pi}{2} - \gamma(k), \frac{\pi}{2} - \gamma(k))$, where $\gamma(k) = \arctan \frac{b-a}{2\sqrt{(k+a)(k+b)}}$ (see Example 3.3 above). Similarly to as in that example, for each $\varepsilon > 0$ and $k > \frac{b-a-(a+b)\sin \varepsilon}{2\sin \varepsilon}$, the composition semigroup generated by Γ^k has an analytic extension to the sector $\Omega(\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} - \varepsilon)$.*

Corollary 4.3. *Let \mathbf{T} be a C_0 -semigroup of composition operators on $H^p(\Pi)$ induced by a semigroup $\mathbf{F} \subset \text{Hol}(\Pi)$. If \mathbf{T} can be analytically extended to sector, then \mathbf{F} is of parabolic type.*

Proof. Since the elements of \mathbf{T} are bounded, the elements of \mathbf{F} have finite angular derivatives at ∞ . Therefore, the generator f of \mathbf{F} has also a finite angular derivative δ at ∞ , and this derivative is known to be real (cf. [26]). It follows by results in [4] and Theorem 4.1 above that there exists small enough $\theta > 0$ such that the function $f_\theta := e^{i\theta} f$ is also a semigroup generator. The angular derivative of f_θ at ∞ equals $e^{i\theta} \delta$, which is real only if $\delta = 0$. So, \mathbf{F} is of parabolic type. \square

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